From kinetic theory to dissipative fluid dynamics

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Abstract

We present the results of deriving the Israel-Stewart equations of relativistic dissipative fluid dynamics from kinetic theory via Grad's 14-moment expansion. Working consistently to second order in the Knudsen number, these equations contain several new terms which are absent in previous treatments.

1 Introduction

Fluid dynamics has always been an important qualitative tool to describe the collective flow of hot and dense matter created in heavy-ion collisions [1]. However, the great success of the fluid dynamical model only came with nuclear collisions at RHIC energies, where for the first time fluid dynamics was able to describe flow observables, such as the elliptic flow [2], on a *quantitative* level [3].

The respective calculations [3] of the elliptic flow at RHIC energies were based on *ideal* fluid dynamics, i.e., all *dissipative* effects were neglected. The good agreement between data and the ideal fluid dynamical calculations gave rise to the notion that "RHIC scientists serve up the perfect fluid". This implies that dissipative effects have to be small in order to not spoil the agreement with data.

However, the fluid dynamical equations of motion are partial differential equations which require initial conditions in order to solve them uniquely. In principle, these introduce infinitely many degrees of freedom that can be tuned to achieve agreement with the data (once an appropriate "freeze-out" prescription is adopted to convert the fluid into particles [4]). Since the initial conditions in heavy-ion collisions are not known sufficiently precisely, one has to perform calculations within relativistic *dissipative* fluid dynamics and for *various realistic* initial conditions in order to confirm the smallness of dissipative effects.

Unfortunately, a consistent, stable, and causal formulation of relativistic dissipative fluid dynamics is far from trivial. The so-called *first-order* theories due to Eckart [5] and Landau [6] have been shown [7] to lead to unstable solutions and to support acausal propagation of perturbations. A viable candidate for a relativistic formulation of dissipative fluid dynamics, which does not have these problems (for a large class of equations of state [8]), is the so-called *second-order* theory due to Israel and Stewart [9]. In recent years, theoreticians have begun to apply this theory for the description of collective flow in heavy-ion collisions [10, 11, 12].

In this paper we present the results of an analysis (the details of which will be presented elsewhere [13]) of deriving the Israel-Stewart equations of relativistic dissipative fluid dynamics from kinetic theory, using the Boltzmann equation and Grad's 14-moment [14]. We show that, working consistently to second order in the Knudsen number, additional terms arise in these equations that have been missed in the original paper [9] and in subsequent treatments [10, 11, 12].

This paper is organized as follows. After introducing our notation in Sec. 2, we discuss the power counting scheme in terms of the Knudsen number (Sec. 3). In Sec. 4 we present and discuss the Israel-Stewart equations. A concluding section summarizes our results and gives an outlook to future work. Our units are $\hbar = c = k_B = 1$, the metric tensor is $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$. The scalar product of 4-vectors A^μ, B^μ is denoted as $A^\mu g_{\mu\nu} B^\nu = A^\mu B_\mu \equiv A \cdot B$.

2 Preliminaries

The quantities that are evolved through the fluid dynamical equations are the 4-current of net charge, N^μ (for the sake of simplicity, we only consider a single conserved charge, the generalization to several charges is straightforward [15]), and the symmetric rank-2 energy-momentum tensor, $T^{\mu\nu}$. In the following, we first discuss the tensor decomposition of these quantities and then give the corresponding fluid dynamical equations which express the conservation of net charge and energy-momentum. Finally, the Navier-Stokes approximation to relativistic dissipative fluid dynamics is discussed.

2.1 *Tensor Decomposition of N^μ and $T^{\mu\nu}$*

The tensor decomposition of the net charge 4-current with respective to an arbitrary 4-vector u^μ reads

$$N^\mu = n\, u^\mu + \nu^\mu \,. \tag{1}$$

We shall identify u^μ with the *fluid 4-velocity* which is time-like and normalized to one, $u \cdot u = 1$. This yields $u^\mu = \gamma(1, \mathbf{v})$, where $\mathbf{v}$ is the fluid 3-velocity and $\gamma = (1-\mathbf{v}^2)^{-1/2}$ the corresponding Lorentz gamma factor. Due to the normalization condition, the 4-vector u^μ has only three independent components. The component of N^μ in the direction of u^μ is the *net charge density* in the fluid rest frame, $n \equiv N \cdot u$. The component orthogonal to u^μ is the *diffusion current*, i.e., the flow of net charge relative to u^μ, $\nu^\mu \equiv \Delta^{\mu\nu} N_\nu$. Here, $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ denotes the projector onto the 3-space orthogonal to u^μ. By construction, $\Delta^{\mu\nu} u_\nu = 0$, and thus also $\nu \cdot u = 0$. Therefore, the 4-vector ν^μ has only three independent components.

The tensor decomposition of the energy-momentum tensor reads

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu - (p + \Pi)\, \Delta^{\mu\nu} + 2\, q^{(\mu} u^{\nu)} + \pi^{\mu\nu} \,, \tag{2}$$

where $\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$ is the *energy density* in the fluid rest frame. The projection $p + \Pi \equiv -\frac{1}{3}\, \Delta^{\mu\nu} T_{\mu\nu}$ is the sum of *thermodynamic pressure*, p, and *bulk viscous pressure*, Π. The *heat flux current*, i.e., the flow of energy relative to u^μ, is $q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$. By construction, $q \cdot u = 0$, and q^μ has only three independent components. The notation $a^{(\alpha_1 \cdots \alpha_n)}$ stands for symmetrization in all Lorentz indices, e.g., $a^{(\mu\nu)} \equiv \frac{1}{2}\,(a^{\mu\nu} + a^{\nu\mu})$. Finally, $\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ is the *shear stress tensor*, where $a^{<\mu\nu>} \equiv \left(\Delta_\alpha{}^{(\mu} \Delta^{\nu)}{}_\beta - \frac{1}{3}\, \Delta^{\mu\nu} \Delta_{\alpha\beta}\right) a^{\alpha\beta}$ denotes the symmetrized, traceless spatial projection. Thus, by construction, $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = \pi^\mu{}_\mu = 0$. This implies that $\pi^{\mu\nu}$ has only five independent components.

2.2 *Fluid Dynamical Equations of Motion*

The conservation of net charge reads

$$\partial \cdot N = \dot{n} + n\, \theta + \partial \cdot \nu = 0 \,, \tag{3}$$

where $\dot{a} \equiv u \cdot \partial\, a$ denotes the comoving derivative. In the fluid rest frame, $u^\mu_{\rm RF} = (1,0,0,0)$, it is simply the time derivative, $\dot{a}_{\rm RF} = \partial_t a$. The quantity $\theta \equiv \partial \cdot u$ is the so-called *expansion scalar*.

The equation for energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$ is a 4-vector and can also be tensor-decomposed into a component parallel to u^μ,

$$u_\nu\, \partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\, \theta + \partial \cdot q - q \cdot \dot{u} - \pi^{\mu\nu}\, \partial_\mu u_\nu = 0\;, \tag{4}$$

which represents the *conservation of energy*, and into the three independent components orthogonal to u^μ, $\Delta^{\mu\nu}\, \partial^\lambda T_{\nu\lambda} = 0$, which can be cast into the form

$$(\epsilon + p)\, \dot{u}^\mu = \nabla^\mu (p + \Pi) - \Pi\, \dot{u}^\mu - \Delta^{\mu\nu} \dot{q}_\nu - q^\mu\, \theta - q \cdot \partial\, u^\mu - \Delta^{\mu\nu}\, \partial^\lambda \pi_{\nu\lambda}\;, \tag{5}$$

where the notation $\nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu$ stands for the 3-gradient, i.e., the spatial gradient in the fluid rest frame. Equation (5) is the so-called *acceleration equation*. The first term on the r.h.s. expresses the fact that changes in the fluid 4-velocity are driven by pressure gradients.

The problem with the five equations of motion (3), (4), and (5) is that, for any given fluid 4-velocity u^μ, they contain 15 unknowns, ϵ, p, n, Π, the three components of ν^μ, the three components of q^μ, and the five components of $\pi^{\mu\nu}$. The choice of reference frame does not alleviate this problem: in the Eckart frame $\nu^\mu \equiv 0$, and in the Landau frame $q^\mu \equiv 0$, however, then u^μ is no longer fixed, but becomes a dynamical quantity, and one still has 15 unknowns. In principle, there are two strategies to continue.

1. One assumes the fluid to be in *local thermodynamical equilibrium*. Then, all dissipative quantities vanish, $\Pi = q^\mu$ (or ν^μ) $= \pi^{\mu\nu} \equiv 0$, which eliminates nine out of the 15 unknowns. This is the so-called *ideal-fluid limit*. In this case, the reference frame is uniquely defined. The sixth remaining unknown, for instance p, is determined by choosing an equation of state for the fluid in the form $p = p(\epsilon, n)$. Then, the system of five equations (3), (4), and (5) can be uniquely solved for ϵ, n, and u^μ.

2. One provides equations that determine the dissipative quantities: the equations of *dissipative fluid dynamics*. One distinguishes *first-order* and *second-order theories* of dissipative fluid dynamics. A first-order theory is, e.g., the *Navier-Stokes* (NS) approximation where the dissipative quantities Π, q^μ (or ν^μ), and $\pi^{\mu\nu}$ are expressed solely in terms of the primary variables ϵ, p, n, u^μ, or gradients thereof. A second-order theory is, for instance, the so-called *Israel-Stewart* (IS) theory [9]. Here, the dissipative quantities are independent dynamical quantities whose evolution is governed by differential equations similar to the fluid dynamical equations (3), (4), and (5).

2.3 *Navier-Stokes Approximation*

In the NS approximation, the dissipative quantities Π, q^μ, $\pi^{\mu\nu}$ read:

$$\Pi_{\rm NS} = -\zeta\, \theta\;, \tag{6}$$

$$q^\mu_{\rm NS} = \frac{\kappa}{\beta}\, \frac{n}{\beta(\epsilon + p)}\, \nabla^\mu \alpha\;, \tag{7}$$

$$\pi^{\mu\nu}_{\rm NS} = 2\, \eta\, \sigma^{\mu\nu}\;, \tag{8}$$

where $\beta \equiv 1/T$ and $\alpha \equiv \beta\mu$; μ is the *chemical potential* associated with the net charge density n. The quantitites ζ, κ, and η are the *bulk viscosity*, *thermal conductivity*, and *shear viscosity* coefficients. The *shear tensor* is defined as $\sigma^{\mu\nu} \equiv \nabla^{<\mu} u^{\nu>}$. Note that Eq. (7) holds in the Eckart frame; in the Landau frame, simply replace $q^\mu \to -\nu^\mu (\epsilon + p)/n$.

Since Π, q^μ (or ν^μ), and $\pi^{\mu\nu}$ are solely given in terms of the primary variables, one can simply insert the expressions (6), (7), and (8) into the Eqs. (3), (4), and (5) and obtain a closed set of equations of motion: the relativistic generalization of the non-relativistic NS equations. As mentioned in the Introduction, the problem with these equations is that they lead to unstable solutions and support acausal propagation of perturbations.

3 Power counting

In order to derive the IS equations, we first have to discuss the length scales appearing in fluid dynamics. Then we identify the Knudsen number as the small quantity in terms of which one can do consistent power counting. The IS equations will then emerge at second order in an expansion in powers of the Knudsen number.

3.1 *Scales in fluid dynamics*

In principle, there are three length scales in fluid dynamics, two microscopic scales and one macroscopic scale. The two microscopic scales are the *thermal wavelength*, $\lambda_{\rm th} \sim \beta$, and the *mean free path*, $\ell_{\rm mfp} \sim (\langle\sigma\rangle n)^{-1}$, where $\langle\sigma\rangle$ is the average cross section. The macroscopic scale, $L_{\rm hydro}$, is the scale over which the fluid fields ϵ, $n, u^\mu, \ldots$ vary, i.e., gradients of these fields are typically of order $\partial_\mu \sim L_{\rm hydro}^{-1}$. Note that, since $n^{-1/3} \sim \beta \sim \lambda_{\rm th}$, the thermal wavelength can be interpreted as the interparticle distance. Note also that the notion of a mean free path requires the existence of well-defined quasi-particles. In strongly coupled theories, the quasi-particle concept is no longer valid. In this case, there are only two scales, $\lambda_{\rm th}$ and $L_{\rm hydro}$.

As a first important result let us note that the ratios of the transport coefficients ζ, κ/β, and η to the entropy density are solely determined by the ratio of the two microscopic length scales, $\ell_{\rm mfp}/\lambda_{\rm th}$. We demonstrate this explicitly for the shear viscosity to entropy density ratio, similar arguments also hold for the other transport coefficients. For the proof, note that $\eta \sim (\langle\sigma\rangle\lambda_{\rm th})^{-1}$ and $n \sim T^3 \sim s$. Then,

$$\frac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \frac{1}{\langle\sigma\rangle n}\,\frac{1}{\lambda_{\rm th}} \sim \frac{1}{\langle\sigma\rangle\lambda_{\rm th}}\,\frac{1}{n} \sim \frac{\eta}{s}\,. \tag{9}$$

There are two limiting cases, (a) the *dilute-gas limit*, $\ell_{\rm mfp}/\lambda_{\rm th} \sim \eta/s \to \infty$, and (b) the *ideal-fluid limit*, $\ell_{\rm mfp}/\lambda_{\rm th} \sim \eta/s \to 0$. Estimating $\ell_{\rm mfp} \sim \langle\sigma\rangle^{-1}\lambda_{\rm th}^3$, the first case corresponds to $\langle\sigma\rangle/\lambda_{\rm th}^2 \to 0$, i.e., the interaction cross section is much smaller than the area given by the thermal wavelength. In other words, the average distance between collisions is much larger than the interparticle distance. In this sense, the dilute-gas limit can be interpreted as a *weak-coupling limit*. Similarly, the ideal-fluid limit corresponds to $\langle\sigma\rangle/\lambda_{\rm th}^2 \to \infty$. This is the somewhat academic case when interactions happen on a scale much smaller than the interparticle distance. In this sense, this is the limit of *infinite coupling*, i.e., the interactions are so strong that the fluid assumes locally and instantaneously a state of thermodynamical equilibrium.

For *any* value of η/s (and, analogously, ζ/s and $\kappa/(\beta s)$) between these two limits, the equations of dissipative fluid dynamics may be applied for the description of the system. The situation is particularly interesting for $\ell_{\rm mfp}/\lambda_{\rm th} \sim \eta/s \sim 1$ or, equivalently, $\langle\sigma\rangle \sim \lambda_{\rm th}^2 \sim T^{-2}$. In this case, there is only a *single* microscopic scale $\lambda_{\rm th}$ in the problem. This occurs, for instance, in strongly coupled theories without well-defined quasi-particles.

One may derive the equations of dissipative fluid dynamics in terms of a gradient expansion. However, in order to be able to truncate this expansion after a finite number of terms one has to require that gradients are small or, equivalently, that the *Knudsen number* is small. This is discussed in the following subsection.

3.2 *Expansion in Terms of the Knudsen Number*

The Knudsen number is defined as $K \equiv \ell_{\rm mfp}/L_{\rm hydro}$. Since $L_{\rm hydro}^{-1} \sim \partial_\mu$, an expansion in terms of K is equivalent to a gradient expansion, i.e., an expansion in terms of powers of $\ell_{\rm mfp}\,\partial_\mu$.

We can now establish a second important result: provided that the dissipative quantities Π, q^μ (or ν^μ), and $\pi^{\mu\nu}$ do not differ too much from their NS values, the ratios of these quantities to the energy

density are proportional to the Knudsen number. We demonstrate this explicitly for the bulk viscous pressure; similar arguments apply for the heat flux current and the shear stress tensor. We use the fundamental relation of thermodynamics, $\epsilon + p = Ts + \mu n$, to estimate $\beta\,\epsilon \sim \lambda_{\rm th}\,\epsilon \sim s$ and we employ Eq. (6) to write

$$\frac{\Pi}{\epsilon} \sim \frac{\Pi_{\rm NS}}{\epsilon} \sim \frac{\zeta\,\theta}{\epsilon} \sim \frac{\zeta}{\lambda_{\rm th}\,\epsilon}\,\lambda_{\rm th}\,\theta \sim \frac{\zeta}{s}\,\frac{\lambda_{\rm th}}{\ell_{\rm mfp}}\,\ell_{\rm mfp}\,\partial_\mu u^\mu \sim \frac{\zeta}{s}\left(\frac{\ell_{\rm mfp}}{\lambda_{\rm th}}\right)^{-1} K\,u^\mu \sim K\;. \tag{10}$$

In the last step, we have employed Eq. (9) and the fact that $u^\mu \sim 1$. The result is remarkable in the sense that Π/ϵ is *only* proportional to K, and *independent* of the ratio of viscosity to entropy density. The reason is that this ratio drops out on account of Eq. (9).

As a corollary to Eq. (10), we conclude that if the Knudsen number is small, $K \sim \delta \ll 1$, the dissipative quantities are small compared to the primary variables. The system is close to local thermodynamical equilibrium or, in other words, close to the ideal-fluid limit. The equations of dissipative fluid dynamics can then be systematically and in a well-controlled manner derived in terms of a gradient expansion or, equivalently, in terms of a power series in K or, equivalently because of Eq. (10), in terms of powers of dissipative quantities. At zeroth order in K, one obtains the equations of ideal fluid dynamics. At first order in K, one obtains the NS equations. At second order in K, the IS equations emerge.

Finally note that the independence of the ratio of dissipative quantities to primary variables from the viscosity to entropy density ratio has important phenomenological consequences. It guarantees that, provided that gradients of the macroscopic fluid fields (and, thus, K) are sufficiently small, the NS equations are still valid and applicable for the description of systems with large η/s, e.g. water at room temperate and atmospheric pressure.

4 The Israel-Stewart Equations

The IS equations of relativistic dissipative fluid dynamics can be derived from the Boltzmann equation via Grad's 14-moment method [14]. The details of this derivation will be presented elsewhere [13]. To second order in dissipative quantities (or equivalently, because of Eq. (10), to second order in K) the equations read:

$$\begin{aligned}
\Pi &= \Pi_{\rm NS} - \tau_\Pi\,\dot{\Pi} \\
&+ \tau_{\Pi q}\,q\cdot\dot{u} - \ell_{\Pi q}\,\partial\cdot q - \zeta\,\hat{\delta}_0\,\Pi\,\theta \\
&+ \lambda_{\Pi q}\,q\cdot\nabla\alpha + \lambda_{\Pi\pi}\,\pi^{\mu\nu}\sigma_{\mu\nu}\;, \qquad (11)\\
q^\mu &= q^\mu_{\rm NS} - \tau_q\,\Delta^{\mu\nu}\dot{q}_\nu \\
&- \tau_{q\Pi}\,\Pi\,\dot{u}^\mu - \tau_{q\pi}\,\pi^{\mu\nu}\,\dot{u}_\nu + \ell_{q\Pi}\,\nabla^\mu\Pi - \ell_{q\pi}\,\Delta^{\mu\nu}\,\partial^\lambda\pi_{\nu\lambda} + \tau_q\,\omega^{\mu\nu}\,q_\nu - \frac{\kappa}{\beta}\,\hat{\delta}_1\,q^\mu\,\theta \\
&- \lambda_{qq}\,\sigma^{\mu\nu}\,q_\nu + \lambda_{q\Pi}\,\Pi\,\nabla^\mu\alpha + \lambda_{q\pi}\,\pi^{\mu\nu}\,\nabla_\nu\alpha\;, \qquad (12)\\
\pi^{\mu\nu} &= \pi^{\mu\nu}_{\rm NS} - \tau_\pi\,\dot{\pi}^{<\mu\nu>} \\
&+ 2\,\tau_{\pi q}\,q^{<\mu}\dot{u}^{\nu>} + 2\,\ell_{\pi q}\,\nabla^{<\mu}q^{\nu>} + 2\,\tau_\pi\,\pi_\lambda^{\;<\mu}\omega^{\nu>\lambda} - 2\,\eta\,\hat{\delta}_2\,\pi^{\mu\nu}\,\theta \\
&- 2\,\tau_\pi\,\pi_\lambda^{\;<\mu}\sigma^{\nu>\lambda} - 2\,\lambda_{\pi q}\,q^{<\mu}\nabla^{\nu>}\alpha + 2\,\lambda_{\pi\Pi}\,\Pi\,\sigma^{\mu\nu}\;, \qquad (13)
\end{aligned}$$

where $\omega^{\mu\nu} \equiv \frac{1}{2}\,\Delta^{\mu\alpha}\Delta^{\nu\beta}\,(\partial_\alpha u_\beta - \partial_\beta u_\alpha)$ is the *vorticity*. We now discuss these equations.

1. The transport coefficients ζ, κ, η, the relaxation times τ_Π, τ_q, τ_π, the coefficients $\tau_{\Pi q}$, $\tau_{q\Pi}$, $\tau_{q\pi}$, $\tau_{\pi q}$, $\ell_{\Pi q}$, $\ell_{q\Pi}$, $\ell_{q\pi}$, $\ell_{\pi q}$, $\lambda_{\Pi q}$, $\lambda_{\Pi\pi}$, λ_{qq}, $\lambda_{q\Pi}$, $\lambda_{q\pi}$, $\lambda_{\pi q}$, $\lambda_{\pi\Pi}$ are (complicated) functions of α, β, divided by tensor coefficients of the second moment of the collision integral, for details, see Ref. [13].

As one sends the cross section in the collision integral to infinity, all these coefficients go to zero. Since then also the NS values vanish, one ends up with the trivial solution $\Pi = q^\mu = \pi^{\mu\nu} \equiv 0$ to the IS equations, i.e., one recovers the ideal-fluid limit. This is consistent with the discussion in Sec. 3.1.

2. The coefficients $\hat{\delta}_0$, $\hat{\delta}_1$, $\hat{\delta}_2$ are (complicated) functions of α, β.

3. The form of the equations is invariant of the calculational frame (Eckart, Landau, ...), however, the values of the coefficients are frame-dependent. The obvious reason is that the physical interpretation of the dissipative quantities is frame-dependent. For instance, in the Eckart frame, q^μ is the heat flux current, while in the Landau frame, $q^\mu \equiv -\nu^\mu(\epsilon + p)/n$ is the (negative of the) diffusion current, multiplied by the specific enthalpy. Details are given in Ref. [13].

4. The NS terms (the first terms on the r.h.s.) are of *first order* in K, all other terms are of *second order* in K. Consequently, dropping the latter one obtains the NS equations (6), (7), and (8).

5. The so-called *simplified* IS equations (in the terminology of Ref. [12]) emerge by keeping only the first lines of Eqs. (11), (12), and (13). The resulting equations have the simple interpretation that the dissipative quantities Π, q^μ, and $\pi^{\mu\nu}$ *relax* to their corresponding NS values on time scales τ_Π, τ_q, and τ_π, respectively.

6. Considering the *full* IS equations (11), (12), and (13), for times $t < \tau_i$, $i = \Pi, q, \pi$, the dissipative quantities Π, q^μ, $\pi^{\mu\nu}$ are driven towards their NS values. Once they are reasonably close to these, the first terms on the r.h.s. largely cancel against the l.h.s.. The further evolution, for times $t > \tau_i$, is then determined by the remaining, second-order terms. These terms therefore constitute important corrections for times $t > \tau_i$ and should not be neglected.

7. The first two terms in the second line of Eq. (11), the first five terms in the second line of Eq. (12), and the first three terms in the second line of Eq. (13) were also obtained by Israel and Stewart [9], while the remaining second-order terms were missed or neglected. Presumably, Israel and Stewart made the assumption that second-order terms containing θ, $\sigma^{\mu\nu}$, or $\nabla^\mu \alpha$ are even smaller than suggested by power counting in terms of K. The last two terms in the second line of Eq. (11), the last four terms in the second line of Eq. (12), and the last three terms in the second line of Eq. (13) were also obtained by Muronga [11], while the other second-order terms do not appear in that paper. A possible reason is that the corresponding treatment is based on the phenomenological approach to derive the IS equations and terms that do not generate entropy are absent. The terms in the third line of Eqs. (11), (12), and (13) were neither given by Israel and Stewart [9] nor by Muronga [11] and are thus genuinely new to this paper (with one exception discussed below).

8. If we set $\Pi = q^\mu = 0$ in Eq. (13), the resulting equation for $\pi^{\mu\nu}$ is identical to that found in Ref. [16]. In particular, the first term in the third line was already obtained in that paper, where it appeared in the form $(\lambda_1/\eta^2)\, \pi_\lambda^{\;<\mu} \pi^{\nu>\lambda}$. Using the NS value (8) for $\pi^{\nu\lambda}$, which is admissible because we are computing to second order in K, to this order this is identical to $2(\lambda_1/\eta)\, \pi_\lambda^{\;<\mu} \sigma^{\nu>\lambda}$. By comparison with Eq. (13), we thus get a prediction for the coefficient λ_1 from kinetic theory, $\lambda_1 \equiv \tau_\pi\, \eta$, in agreement with Ref. [16]. Note, however, that this discussion so far neglects additional terms which arise at second order in K when expanding the second moment of the collision integral. (This was already noted in Ref. [16].) This will change the coefficient of the respective term such that it is no longer equal to τ_π. It will therefore also lead to a different result for λ_1. In a recent study [17] a complete calculation was performed.

5 Conclusion

In this paper, we have discussed the full Israel-Stewart (IS) equations of relativistic dissipative fluid dynamics as they emerge from applying Grad's 14-moment expansion to the Boltzmann equation and truncating dissipative effects at second order in the Knudsen number $K = \ell_{\mathrm{mfp}}/L_{\mathrm{hydro}}$. Our treatment is not restricted to the shear stress tensor, but also contains the bulk viscous pressure and the heat flux current. It is thus also applicable to non-conformal systems with non-vanishing net charge density.

We have shown that, in comparison to previous discussions [9, 10, 11], additional second-order terms appear. One of these terms was found already in Ref. [16]. The details of the derivation of the full second-order IS equations will be presented elsewhere [13]. The second-order terms are multiplied with coefficients whose values depend on the calculational frame. Explicit expressions will be reported elsewhere [13]. Future directions of work comprise the generalization to a system of various particle species [15], as well as the numerical implementation and application to modelling the dynamics of heavy-ion collisions.

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